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MIXED BOUNDARY-VALUE PROBLEMS FOR ANISOTROPIC PLATES OF VARIABLE THICKNESS†

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Using the equations of the three-dimensional problem of the theory of thermo-elasticity for an anisotropic body the asymptotic solution of the mixed boundary-value problem for a plate of variable thickness is obtained. It is assumed that the displacement vector is prescribed on one of the face surfaces of the plate, while the conditions of the first, second, or mixed boundary-value problem of the theory of elasticity are given on the other surface. Apart from surface forces, the plate is also subjected to given volume forces and temperature fields. It is proved that the Kirchhoff–Love hypotheses are not applicable to this class of problems. Iterative processes are constructed and recursive formulae are obtained, which enable the components of the stress tensor and displacement vector to be determined with an asymptotic accuracy specified in advance. Several examples are considered which illustrate the effectiveness of the resulting formulae and the asymptotic approach.

The asymptotic method [1–6] has turned out to be especially effective for plates and shells with mixed boundary conditions specified on the face surfaces [7–9]. In particular, it has been proved that the hypotheses of the classical theory of plates and shells are inapplicable to this class of problems. Asymptotic solutions of mixed boundary-value problems were constructed in [7–12] for single-layer and multi-layer anisotropic strips and plates of constant thickness. As we shall see below, the asymptotic method is also effective for solving mixed boundary-value problems involving plates of variable thickness.

1. Consider a thin anisotropic body of variable thickness occupying a domain $\Omega = \{\alpha, \beta, \gamma: \alpha, \beta \in \Sigma, -\varphi_2(\alpha, \beta) \leqslant \gamma \leqslant \varphi_1(\alpha, \beta), \varphi, (\alpha, \beta) > 0\}$, where Σ is a plane inside the body that does not intersect the face surfaces. The latter are given by sufficiently smooth functions $\gamma = \varphi_1(\alpha, \beta) > 0$ and $\gamma = -\varphi_2(\alpha, \beta) < 0$, where α and β are curvilinear coordinate axes on Σ which coincide with the principal directions of anisotropy, and γ is the rectilinear axis perpendicular to Σ at the point (α, β).

It is required to determine the stress–strain state of such a plate of variable thickness when the components

$$u_j(-\varphi_2) = u_j^-, \quad j = \alpha, \beta, \gamma \tag{1.1}$$

of the displacement vector are given on the face surface $\gamma = -\varphi_1(\alpha, \beta)$ and the conditions of the first boundary-value problem of the theory of elasticity

$$\sigma_{j\alpha}\cos(n,\alpha) + \sigma_{j\beta}\cos(n,\beta) + \sigma_{j\gamma}\cos(n,\gamma) = F_{nj}, \quad j = \alpha, \beta, \gamma \tag{1.2}$$

the second boundary-value problem

$$u_j(\varphi_1) = u_j^+, \quad j = \alpha, \beta, \gamma \tag{1.3}$$

or the mixed boundary-value problem

$$\text{(a)}\ \sigma_{j\alpha}\cos(n,\alpha) + \sigma_{j\beta}\cos(n,\beta) + \sigma_{j\gamma}\cos(n,\gamma) = F_{nj}, \quad j = \alpha, \beta$$

$$u_\gamma(\varphi_1) = u_\gamma^+ \tag{1.4}$$

$$\text{(b)}\ u_j(\varphi_1) = u_j^-, \quad j = \alpha, \beta$$

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$$\sigma_{\alpha\gamma}\cos(n,\alpha)+\sigma_{\beta\gamma}\cos(n,\beta)+\sigma_{\gamma\gamma}\cos(n,\gamma)=F_{n\gamma} \tag{1.5}$$

are specified on the opposite surface $\gamma = -\varphi_1(\alpha, \beta)$, where $F_{nj}(\alpha, \beta)$, $u_j^{\mp}(\alpha, \beta)$ $(j = \alpha, \beta, \gamma)$ are given functions and

$$\cos(n,\alpha)=-\frac{1}{\lambda A}\frac{\partial\varphi_1}{\partial\alpha} \quad (\alpha,\beta;A,B) \tag{1.6}$$

$$\cos(n,\gamma)=\frac{1}{\lambda}, \quad \lambda=\left[1+\left(\frac{1}{A}\frac{\partial\varphi_1}{\partial\alpha}\right)^2+\left(\frac{1}{B}\frac{\partial\varphi_1}{\partial\beta}\right)^2\right]^{1/2}$$

where A and B are the Lamé coefficients of the first quadratic form.

We assume that the plate is acted upon by volume forces $\mathbf{P} = \{P_\alpha, P_\beta, P_\gamma\}$ and temperature fields conforming to the Duhamel–Neumann model.

In the equilibrium equations and elasticity relationships we change to dimensionless coordinates and dimensionless displacements using the formulae

$$\begin{gathered}\xi=\alpha/a, \quad \eta=\beta/a, \quad \zeta=\gamma/h=\varepsilon^{-1}\gamma/a \\ u_\alpha=au, \quad u_\beta=av, \quad u_\gamma=aw \\ (h=\max\{\sup\varphi_1(\alpha,\beta),\ \sup\varphi_2(\alpha,\beta)\},\ \varepsilon=h/a)\end{gathered} \tag{1.7}$$

where a is the characteristic dimension of the plate in the target plane, ε is a small parameter and $a \gg h$.

As a result, we have the system

$$\begin{gathered}\frac{1}{A}\frac{\partial\sigma_{\alpha\alpha}}{\partial\xi}+\frac{1}{B}\frac{\partial\sigma_{\alpha\beta}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{\alpha\gamma}}{\partial\zeta}+a(\sigma_{\alpha\alpha}-\sigma_{\beta\beta})k_\beta+2ak_\alpha\sigma_{\alpha\beta}+aP_\alpha=0 \quad (\alpha,\beta;\xi,\eta;A,B) \\ \frac{1}{A}\frac{\partial\sigma_{\alpha\gamma}}{\partial\xi}+\frac{1}{B}\frac{\partial\sigma_{\beta\gamma}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{\gamma\gamma}}{\partial\zeta}+ak_\beta\sigma_{\alpha\gamma}+ak_\alpha\sigma_{\beta\gamma}+aP_\gamma=0 \\ \frac{1}{A}\frac{\partial u}{\partial\xi}+ak_\alpha v=a_{11}\sigma_{\alpha\alpha}+a_{12}\sigma_{\beta\beta}+a_{13}\sigma_{\gamma\gamma}+a_{14}\sigma_{\beta\gamma}+a_{15}\sigma_{\alpha\gamma}+a_{16}\sigma_{\alpha\beta}+\alpha_{11}\theta \\ (\alpha,\beta;\xi,\eta;u,v;A,B;1,2) \\ \varepsilon^{-1}\frac{\partial w}{\partial\zeta}=a_{13}\sigma_{\alpha\alpha}+a_{23}\sigma_{\beta\beta}+\ldots+a_{36}\sigma_{\alpha\beta}+\alpha_{33}\theta \\ \frac{1}{B}\frac{\partial u}{\partial\eta}+\frac{1}{A}\frac{\partial v}{\partial\xi}-a(k_\alpha u+k_\beta v)=a_{16}\sigma_{\alpha\alpha}+\ldots+a_{66}\sigma_{\alpha\beta}+\alpha_{12}\theta \\ \frac{1}{A}\frac{\partial w}{\partial\xi}+\varepsilon^{-1}\frac{\partial u}{\partial\zeta}=a_{15}\sigma_{\alpha\alpha}+\ldots+a_{55}\sigma_{\alpha\gamma}+a_{56}\sigma_{\alpha\beta}+\alpha_{13}\theta \\ (\xi,\eta;\ u,v;\ 5,4;\ \alpha_{13},\alpha_{23})\end{gathered} \tag{1.8}$$

where α_{jk} are the coefficients of thermal expansion and $\theta = T - T_0$ is the temperature increment.

In (1.8) the variability with respect to the coordinates could be taken into account by using the dimensionless coordinates introduced in [1], rather than (1.7). In the problems under consideration the solutions are obtained in the form of complete formulae, which implies that the influence of variability is one of the factors determining a common estimate of the remainder.

We shall seek a solution of (1.8) as an asymptotic expansion

$$Q=\sum_{s=0}^{N}\varepsilon^{\varkappa_a+s}Q^{(s)}(\xi,\eta,\zeta) \tag{1.9}$$

where Q is any of the quantities to be determined, $\varkappa_\sigma = -1$ for stresses, and $\varkappa_u = 0$ for strains. For the contribution of the volume forces and the temperature field to be commensurable with that of the surface interactions it is necessary that

$$P_j = \sum_{s=0}^{N} \varepsilon^{-2+s} a^{-1} P_j^{(s)}, \quad \theta = \sum_{s=0}^{N} \varepsilon^{-1+s} \theta^{(s)}(\xi,\eta,\zeta) \tag{1.10}$$

The asymptotic form (1.9) differs in a major way from that of the same quantities in the classical theory of plates. For example, here all the stresses have the same asymptotic order, whereas they are different in the classical theory [2, 5]. Also, we shall find that only in the case of asymptotic expansion (1.9) is it possible to obtain a non-contradictory system for the unknown coefficients $Q^{(s)}$ of the expansion.

Substituting (1.9) and (1.10) into (1.8) and comparing the coefficients of like powers of ε on both sides of each of the equations in (1.8), we obtain a non-contradictory system for the quantities $Q^{(s)}(\xi, \eta, \zeta)$ in (1.9). By solving the resulting system and satisfying the boundary conditions (1.1), we obtain the following recursive formulae, from which to determine the stresses and strains

$$\sigma_{j\gamma}^{(s)} = \sigma_{j\gamma 0}^{(s)}(\xi,\eta) + \sigma_{j\gamma *}^{(s)}(\xi,\eta,\zeta), \quad j = \alpha,\beta,\gamma$$

$$\sigma_{\alpha\alpha}^{(s)} = A_{13}\sigma_{\gamma\gamma 0}^{(s)} + A_{14}\sigma_{\beta\gamma 0}^{(s)} + A_{15}\sigma_{\alpha\gamma 0}^{(s)} + \sigma_{\alpha\alpha *}^{(s)}(\xi,\eta,\zeta)$$

$$(\alpha\alpha,\ \beta\beta,\ \alpha\beta;\ 1,\ 2,\ 6)$$

$$u^{(s)} = (\zeta+\zeta_2)(A_{53}\sigma_{\gamma\gamma 0}^{(s)} + A_{54}\sigma_{\beta\gamma 0}^{(s)} + A_{55}\sigma_{\alpha\gamma 0}^{(s)} + u_*^{(s)}(\zeta) - u_*^{(s)}(-\zeta_2) + u^{-(s)}, \quad \zeta_2 = h^{-1}\varphi_2$$

$$(u,\ v,\ w;\ 5l,\ 4l,\ 3l;\ l=3,4,5)$$

$$u^{-(0)} = u_\alpha^-, \quad u^{-(s)} = 0, \quad s \neq 0 \quad (u,\ v,\ w;\ \alpha,\ \beta,\ \gamma)$$

$$\sigma_{\alpha\gamma *}^{(s)} = -\int_0^\zeta \left[\frac{1}{A}\frac{\partial \sigma_{\alpha\alpha}^{(s-1)}}{\partial \xi} + \frac{1}{B}\frac{\partial \sigma_{\alpha\beta}^{(s-1)}}{\partial \eta} + ak_\beta(\sigma_{\alpha\alpha}^{(s-1)} - \sigma_{\beta\beta}^{(s-1)}) + 2ak_\alpha\sigma_{\alpha\beta}^{(s-1)} + P_\alpha^{(s)} \right] d\zeta \quad (\alpha,\beta;\xi,\eta;A,B)$$

$$\sigma_{\gamma\gamma *}^{(s)} = -\int_0^\zeta \left[\frac{1}{A}\frac{\partial \sigma_{\alpha\gamma}^{(s-1)}}{\partial \xi} + \frac{1}{B}\frac{\partial \sigma_{\beta\gamma}^{(s-1)}}{\partial \eta} + ak_\beta(\sigma_{\alpha\gamma}^{(s-1)} + ak_\alpha\sigma_{\beta\gamma}^{(s-1)} + P_\gamma^{(s)} \right] d\zeta$$

$$\sigma_{\alpha\alpha *}^{(s)} = B_{11}R_1^{(s)} + B_{12}R_2^{(s)} + B_{16}R_3^{(s)} \tag{1.11}$$

$$(\alpha\alpha,\ \beta\beta,\ \alpha\beta;\ 1l,\ 2l,\ 6l;\ l=1,2,6)$$

$$u_*^{(s)} = \int_0^\zeta (a_{15}\sigma_{\alpha\alpha *}^{(s)} + a_{25}\sigma_{\beta\beta *}^{(s)} + \ldots + a_{65}\sigma_{\alpha\beta *}^{(s)} - \frac{1}{A}\frac{\partial w^{(s-1)}}{\partial \xi} + \alpha_{13}\theta^{(s)})d\zeta$$

$$(u,\ v;\ \xi,\ \eta;\ 5,\ 4;\ A,\ B;\ 13;\ 23)$$

$$w_*^{(s)} = \int_0^\zeta (a_{13}\sigma_{\alpha\alpha *}^{(s)} + a_{23}\sigma_{\beta\beta *}^{(s)} + \ldots + a_{36}\sigma_{\alpha\beta *}^{(s)} + \alpha_{33}\theta^{(s)})d\zeta$$

$$R_1^{(s)} = \frac{1}{A}\frac{\partial u^{(s-1)}}{\partial \xi} + ak_\alpha v^{(s-1)} - a_{13}\sigma_{\gamma\gamma *}^{(s)} - a_{14}\sigma_{\beta\gamma *}^{(s)} - a_{15}\sigma_{\alpha\gamma *}^{(s)} - \alpha_{11}\theta^{(s)}$$

$$(\xi,\ \eta;\ u,\ v;\ A,\ B;\ k_\alpha,\ k_\beta;\ 1,\ 2)$$

$$R_3^{(s)} = \frac{1}{B}\frac{\partial u^{(s-1)}}{\partial \eta} + \frac{1}{A}\frac{\partial v^{(s-1)}}{\partial \xi} - a(k_\alpha u^{(s-1)} + k_\beta v^{(s-1)}) - a_{36}\sigma_{\gamma\gamma *}^{(s)} - a_{46}\sigma_{\beta\gamma *}^{(s)} - a_{56}\sigma_{\alpha\gamma *}^{(s)} - \alpha_{12}\theta^{(s)}$$

$$B_{mn} = (a_{mk}a_{nk} - a_{mn}a_{kk})\Delta^{-1}, \quad B_{kk} = (a_{mm}a_{nn} - a_{mn}^2)\Delta^{-1}$$

$$B_{mn} = B_{nm}, \quad m \neq n \neq k \neq m, \quad m,n,k = 1,2,6$$

$$A_{ml} = -a_{1l}B_{m1} - a_{2l}B_{m2} - a_{6l}B_{m6}$$

$$A_{pl} = a_{p1}A_{1l} + a_{p2}A_{2l} + a_{p6}A_{6l} + a_{pl}, \quad A_{pl} \neq A_{lp}, \quad p,l = 3,4,5$$

$$\Delta = a_{11}a_{22}a_{66} + 2a_{12}a_{16}a_{26} - a_{11}a_{26}^2 - a_{22}a_{16}^2 - a_{66}a_{12}^2$$

The solution (1.11) contains the integration functions $\sigma^{(s)}_{\alpha\gamma 0}$, $\sigma^{(s)}_{\beta\gamma 0}$, $\sigma^{(s)}_{\gamma\gamma 0}$, which are so far unknown but can be uniquely determined from the boundary conditions on the face surface $\gamma = \varphi_1(\alpha, \beta)$.

1. If the coefficients of the stress tensor (1.2) are given on the face surface $\gamma = \varphi_1(\alpha, \beta)$, we obtain

$$
\begin{aligned}
&\sigma^{(s)}_{\alpha\gamma 0} = (D^{33}_{24}\Phi^{(s)}_{\alpha} + D^{34}_{13}\Phi^{(s)}_{\beta} + D^{23}_{14}\Phi^{(s)}_{\gamma})\Delta_*^{-1}\\
&\sigma^{(s)}_{\beta\gamma 0} = (D^{35}_{23}\Phi^{(s)}_{\alpha} + D^{33}_{15}\Phi^{(s)}_{\beta} + D^{25}_{13}\Phi^{(s)}_{\gamma})\Delta_*^{-1}\\
&\sigma^{(s)}_{\gamma\gamma 0} = (D^{34}_{25}\Phi^{(s)}_{\alpha} + D^{35}_{14}\Phi^{(s)}_{\beta} + D^{24}_{15}\Phi^{(s)}_{\gamma})\Delta_*^{-1}\\
&D^{jl}_{ik} = C_{ik}C_{jl} - C_{il}C_{jk}, \quad \Delta_* = C_{33}D^{24}_{15} + C_{34}D^{25}_{13} + C_{35}D^{23}_{14} \neq 0\\
&\Phi^{(s)}_{j} = \lambda F^{(s)}_{nj} + \sigma^{(s)}_{j\alpha *}(\gamma_1)\psi_\alpha + \sigma^{(s)}_{j\beta *}(\gamma_1)\psi_\beta - \sigma^{(s)}_{j\gamma *}(\gamma_1)\\
&F^{(0)}_{nj} = F_{nj}, \quad F^{(s)}_{nj} = 0, \quad s \neq 0, \quad j = \alpha,\beta,\gamma; \quad \gamma_1 = \varphi_1(\alpha,\beta)\\
&C_{13} = -A_{13}\psi_\alpha - A_{63}\psi_\beta, \quad C_{14} = -A_{14}\psi_\alpha - A_{64}\psi_\beta\\
&C_{15} = 1 - A_{15}\psi_\alpha - A_{65}\psi_\beta, \quad C_{23} = -A_{63}\psi_\alpha - A_{23}\psi_\beta\\
&C_{24} = 1 - A_{64}\psi_\alpha - A_{24}\psi_\beta, \quad C_{25} = -A_{65}\psi_\alpha - A_{25}\psi_\beta\\
&C_{33} = 1, \quad C_{34} = -\psi_\beta, \quad C_{35} = -\psi_\alpha\\
&\psi_\alpha = \frac{1}{A}\frac{\partial\varphi_1}{\partial\alpha}, \quad \psi_\beta = \frac{1}{B}\frac{\partial\varphi_1}{\partial\beta}
\end{aligned}
\tag{1.12}
$$

From the recursive formulae (1.9)–(1.12) it follows that in the case of rectilinear anisotropy of the compressible layer, when the surface $\gamma = \varphi_1(\alpha, \beta)$ is a plane and the functions $\gamma = -\varphi_1(\alpha, \beta)$, F_{nj}, u_j^- ($j = \alpha, \beta, \gamma$) the volume forces $\mathbf{P} = \{P_\alpha, P_\beta, P_\gamma\}$, and the temperature function $\theta(\alpha, \beta, \gamma)$ are polynomials of degree no greater than m, the iterative process will terminate after $(m + 1)$ steps and the exact solution of the boundary-value problem for the layer will be obtained [7, 8, 11].

2. If the components of the displacement vector (1.3) are given on the surface $\gamma = \varphi_1(\alpha, \beta)$, we obtain the following expressions for the integration functions $\sigma^{(s)}_{\alpha\gamma 0}$, $\sigma^{(s)}_{\beta\gamma 0}$, $\sigma^{(s)}_{\gamma\gamma 0}$

$$
\begin{aligned}
&\sigma^{(s)}_{\alpha\gamma 0} = D^*_{55}V^{(s)}_\alpha + D^*_{54}V^{(s)}_\beta + D^*_{53}V^{(s)}_\gamma \quad (\alpha\gamma,\ \beta\gamma,\ \gamma\gamma;\ 5l,\ 4l,\ 3l;\ l = 5,4,3)\\
&V^{(s)}_\alpha = \frac{1}{\varphi_1 + \varphi_2}(u^{+(s)} - u^{-(s)} + u^{(s)}_*(-\varphi_2) - u^{(s)}_*(\varphi_1))\\
&u^{\pm(0)} = u^{\pm}_\alpha / a, \quad u^{\pm(s)} = 0, \quad s = 0 \quad (\alpha,\ \beta,\ \gamma;\ u,\ v,\ w)\\
&D^*_{ij} = (A_{kk}A_{ij} - A_{ik}A_{kj})\Delta_2^{-1}, \quad D^*_{kk} = (A_{ij}A_{ji} - A_{ii}A_{jj})\Delta_2^{-1}\\
&i \neq j \neq k \neq i; \quad i,j,k = 3,4,5\\
&\Delta_2 = A_{33}(A_{45}A_{54} - A_{44}A_{55}) + A_{43}(A_{55}A_{34} - A_{54}A_{35}) + A_{53}(A_{44}A_{35} - A_{34}A_{45})
\end{aligned}
\tag{1.13}
$$

3. In the case of the mixed boundary conditions (1.4) we have

$$
\begin{aligned}
&\sigma^{(s)}_{\alpha\gamma 0} = [(A_{33}C_{24} - A_{34}C_{23})\Phi^{(s)}_\alpha + (A_{34}C_{13} - A_{33}C_{14})\Phi^{(s)}_\beta + D^{23}_{14}V^{(s)}_\gamma]\Delta_3^{-1}\\
&\sigma^{(s)}_{\beta\gamma 0} = [(A_{35}C_{23} - A_{33}C_{25})\Phi^{(s)}_\alpha + (A_{33}C_{15} - A_{35}C_{13})\Phi^{(s)}_\beta + D^{25}_{13}V^{(s)}_\gamma]\Delta_3^{-1}\\
&\sigma^{(s)}_{\gamma\gamma 0} = [(A_{34}C_{25} - A_{35}C_{24})\Phi^{(s)}_\alpha + (A_{35}C_{14} - A_{34}C_{15})\Phi^{(s)}_\beta + D^{24}_{15}V^{(s)}_\gamma]\Delta_3^{-1}\\
&\Delta_3 = A_{33}D^{24}_{15} + A_{34}D^{25}_{13} + A_{35}D^{23}_{14} \neq 0
\end{aligned}
\tag{1.14}
$$

4. If the mixed boundary conditions (1.5) are given on the surface $\gamma = \varphi_1(\alpha, \beta)$, we get

$$
\begin{aligned}
&\sigma_{\alpha\gamma 0}^{(s)} = [(A_{44}+A_{43}\psi_\beta)V_\alpha^{(s)} - (A_{54}+A_{53}\psi_\beta)V_\beta^{(s)} + (A_{43}A_{54}-A_{44}A_{53})\Phi_\gamma^{(s)}]\Delta_4^{-1} \\
&\sigma_{\beta\gamma 0}^{(s)} = [-(A_{45}+A_{43}\psi_\alpha)V_\alpha^{(s)} + (A_{55}+A_{53}\psi_\alpha)V_\beta^{(s)} + (A_{45}A_{53}-A_{55}A_{43})\Phi_\gamma^{(s)}]\Delta_4^{-1} \\
&\sigma_{\gamma\gamma 0} = [(A_{44}\psi_\alpha - A_{45}\psi_\beta)V_\alpha^{(s)} - (A_{54}\psi_\alpha - A_{55}\psi_\beta)V_\beta^{(s)} + (A_{44}A_{55}-A_{45}A_{54})\Phi_\gamma^{(s)}]\Delta_4^{-1} \\
&\Delta_4 = A_{55}A_{44} - A_{45}A_{54} + (A_{44}A_{53}-A_{43}A_{54})\psi_\alpha + (A_{55}A_{43}-A_{45}A_{53})\psi_\beta \neq 0
\end{aligned}
\tag{1.15}
$$

for the integration functions.

It follows that the stresses and strains inside an anisotropic plate (layer) of variable thickness with the corresponding boundary conditions (1.1)–(1.5) can be computed using the asymptotic representation (1.9) and recursive formulae (1.10)–(1.15) with any asymptotic accuracy specified in advance. Because the original boundary-value problem is perturbed in a singular way, the resulting solutions may differ considerably from the exact ones only near the lateral surface. This is where the given solution of the inner problem must be supplemented with a consistent solution of the boundary-layer type [13, 14].

To conclude this section we shall investigate the limits of applicability of the above solutions of boundary-value problems (1.1)–(1.5) and (1.9)–(1.15). The asymptotic character of these solutions requires that

$$
Q = \sum_{s=0}^{N} \varepsilon^{\varkappa_Q + s} Q^{(s)} + o(\varepsilon^{\varkappa_Q + N}) \tag{1.16}
$$

In the general case it is difficult to estimate the remainder term in (1.16) for each N because of the multitude of elastic and geometrical characteristics of the plate, functions describing the face surfaces, and boundary conditions given on these surfaces. Provided that the functions $\gamma = \pm\varphi_i(\alpha, \beta)$, $u_j^{\mp}$, F_{nj}, θ have bounded derivatives of any required order, the estimation of the order of magnitude of the first discarded infinitesimal term of the asymptotic expansion does not present a major difficulty in any particular problem, which will be illustrated by a separate example.

2. To illustrate the results we shall consider a number of examples.

1. We have a reservoir whose bottom is made of a rectilinearly orthotropic material. Suppose that the bottom surface is described by the paraboloid of revolution $z = a(x^2 + y^2)/2$ in the chosen reference system, the horizontal plane $z = H$ being the liquid level. We need to determine the stress–strain state across the thickness of the bottom of variable height depending on the hydrostatic water pressure $F_n = \rho g(H - z)$, given that the lower surface of the compressible layer is rigidly fixed at a depth $z = -h$ (i.e. deeper than $z = -h$ the bottom is assumed to be absolutely rigid) (Fig. 1).

Thus we have a compressible orthotropic layer $-h \leqslant z \leqslant \varphi(x, y)$ of variable thickness with boundary conditions

$$
\begin{aligned}
&u_j\ (z=-h) = 0, \quad j = x, y, z \\
&-ax\sigma_{xx}(\varphi) - ay\sigma_{xy}(\varphi) + \sigma_{xz}(\varphi) = axF_n \\
&-ax\sigma_{xy}(\varphi) - ay\sigma_{yy}(\varphi) + \sigma_{yz}(\varphi) = ayF_n \\
&-ax\sigma_{xz}(\varphi) - ay\sigma_{yz}(\varphi) + \sigma_{zz}(\varphi) = -F_n \\
&F_n = \rho g(H - \varphi), \quad \varphi = ar^2/2, \quad r^2 = x^2 + y^2
\end{aligned}
\tag{2.1}
$$

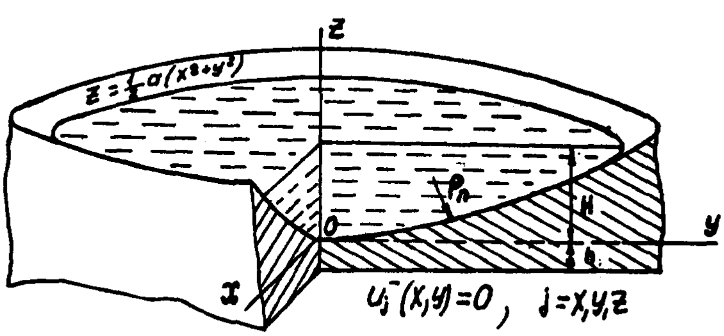

Fig. 1.

where ρ is the liquid density and g is the acceleration due to gravity.

Computing the stresses and strains by (1.11) with the boundary condition (2.1), after the first iteration step we have

$$\sigma_{xx}^{(0)} = A_{13}\sigma_{zz}^{(0)}, \quad \sigma_{xz}^{(0)} = axA_{13}\sigma_{zz}^{(0)} + axF_n, \quad (x,y;1,2) \tag{2.2}$$

$$\sigma_{xy}^{(0)} = 0, \quad \sigma_{zz}^{(0)} = -F_n \frac{1-a^2(x^2+y^2)}{1-a^2x^2A_{13}-a^2y^2A_{23}}$$

$$u_x^{(0)} = (z+h)A_{55}\sigma_{xz}^{(0)} \quad (x,y;5,4), \quad u_z^{(0)} = (z+h)A_{33}\sigma_{zz}^{(0)}$$

The second iteration step yields

$$\sigma_{xz}^{(1)} = f_x + axA_{13}\sigma - zA_{13}\frac{\partial\sigma_{zz}^{(0)}}{\partial x} \quad (x,y;1,2)$$

$$\sigma_{zz}^{(1)} = \sigma - z\left(\frac{\partial\sigma_{xz}^{(0)}}{\partial x} + \frac{\partial\sigma_{yz}^{(0)}}{\partial y}\right), \quad \sigma = \frac{axf_x + ayf_y + f_z}{1-a^2x^2A_{13}-a^2y^2A_{23}}$$

$$\sigma_{xx}^{(1)} = A_{13}\sigma_{zz}^{(1)} + (z+h)\left(B_{11}A_{55}\frac{\partial\sigma_{xz}^{(0)}}{\partial x} + B_{12}A_{44}\frac{\partial\sigma_{yz}^{(0)}}{\partial y}\right)$$

$$\sigma_{yy}^{(1)} = A_{23}\sigma_{zz}^{(1)} + (z+h)\left(B_{12}A_{55}\frac{\partial\sigma_{xz}^{(0)}}{\partial x} + B_{22}A_{44}\frac{\partial\sigma_{yz}^{(0)}}{\partial y}\right)$$

$$\sigma_{xy}^{(1)} = B_{66}(z+h)\left(A_{55}\frac{\partial\sigma_{xz}^{(0)}}{\partial y} + A_{44}\frac{\partial\sigma_{yz}^{(0)}}{\partial x}\right)$$

$$u_x^{(1)} = A_{55}(z+h)(f_x + axA_{13}\sigma) - \frac{1}{2}A_{13}A_{55}(z^2-h^2)\frac{\partial\sigma_{zz}^{(0)}}{\partial x} - \frac{1}{2}A_{33}(z+h)^2\frac{\partial\sigma_{zz}^{(0)}}{\partial x} \quad (x,y;13,23;5,4) \tag{2.3}$$

$$u_z^{(1)} = A_{33}(z+h)\sigma - \frac{1}{2}(z+h)^2\left(A_{13}A_{55}\frac{\partial\sigma_{xz}^{(0)}}{\partial x} - A_{23}A_{44}\frac{\partial\sigma_{yz}^{(0)}}{\partial y}\right) - \frac{1}{2}A_{33}(z^2-h^2)\left(\frac{\partial\sigma_{xz}^{(0)}}{\partial x} + \frac{\partial\sigma_{yz}^{(0)}}{\partial y}\right)$$

$$f_x = ax(\varphi+h)\left(B_{11}A_{55}\frac{\partial\sigma_{xz}^{(0)}}{\partial x} + B_{12}A_{44}\frac{\partial\sigma_{yz}^{(0)}}{\partial y}\right) + B_{66}ay(\varphi+h)\left(A_{55}\frac{\partial\sigma_{xz}^{(0)}}{\partial y} + A_{44}\frac{\partial\sigma_{yz}^{(0)}}{\partial x}\right) +$$

$$+A_{13}\varphi\left[\frac{\partial\sigma_{zz}^{(0)}}{\partial x} - ax\left(\frac{\partial\sigma_{xz}^{(0)}}{\partial x} + \frac{\partial\sigma_{yz}^{(0)}}{\partial y}\right)\right] \quad (x,y;1,2;5,4)$$

$$f_z = \varphi\left(\frac{\partial\sigma_{xz}^{(0)}}{\partial x} + \frac{\partial\sigma_{yz}^{(0)}}{\partial y} - axA_{13}\frac{\partial\sigma_{zz}^{(0)}}{\partial x} - ayA_{23}\frac{\partial\sigma_{zz}^{(0)}}{\partial y}\right)$$

Adding the first two approximations in (2.2) and (2.3), we can determine the components of the stress tensor and the displacement vector to within $O(\varepsilon^2)$, which is, as a rule, sufficient in practical calculations. By continuing the iterative computation, one can obtain more accurate results.

2. Suppose that the face surfaces $z = \varphi_1(x,y)$ and $z = -\varphi_2(x,y)$ of an orthotropic layer (plate) of variable thickness are rigidly fixed. The layer is subject to a temperature field which varies over the transverse coordinate, $\theta = \theta(z)$. It is required to determine the stress–strain state of the layer (Fig. 2). This problem arises, in particular, in the process of high-temperature stamping out of details of a given shape.

The boundary conditions are

$$u_j(z=\varphi_1) = u_j(z=-\varphi_2) = 0, \quad j = x,y,z \tag{2.4}$$

After the first iteration step we obtain

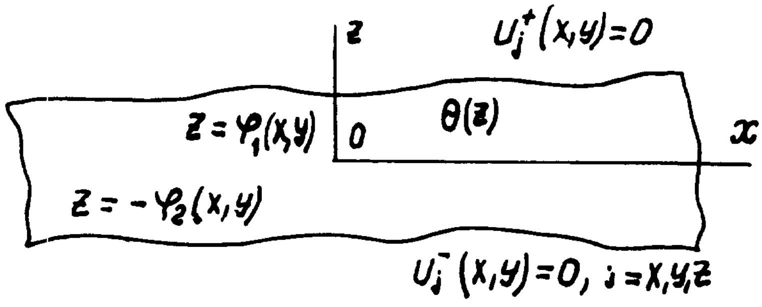

Fig. 2.

$$
\begin{aligned}
&\sigma_{xz}=\frac{\alpha_{13}}{A_{55}}\Phi \quad (x,y;1,2;5,4),\quad \sigma_{zz}=\frac{g_{33}}{A_{33}}\Phi\\
&\sigma_{xx}=A_{13}\frac{g_{33}}{A_{33}}\Phi-(\alpha_{11}B_{11}+\alpha_{22}B_{12})\theta,\quad \sigma_{xy}=-\frac{\alpha_{12}}{a_{66}}\theta\\
&\sigma_{yy}=A_{23}\frac{g_{33}}{A_{33}}\Phi-(\alpha_{11}B_{12}+\alpha_{22}B_{22})\theta\\
&u_x=\alpha_{13}(z+\varphi_2)\Phi+\alpha_{13}(F(z)-F(-\varphi_2)) \quad (x,y;13,23)\\
&u_z=g_{33}(z+\varphi_2)\Phi+g_{33}(F(z)-F(-\varphi_2))\\
&F(z)=\int_0^z\theta dz,\quad \Phi=\frac{1}{\varphi_1+\varphi_2}(F(-\varphi_2)-F(\varphi_1))\\
&g_{33}=\alpha_{11}A_{13}+\alpha_{22}A_{23}+\alpha_{33}
\end{aligned}
\tag{2.5}
$$

with accuracy $O(\varepsilon)$.

To estimate the contribution of the physico-mechanical parameters and the variability of the boundary functions in the next approximation we state the values of the stresses corresponding to the approximation $s = 1$. We have

$$
\begin{aligned}
&\sigma_{xz}^{(1)}=\varepsilon\left(\frac{\varphi_1-\varphi_2}{2h}-\zeta\right)\frac{A_{13}}{A_{33}}g_{33}\frac{\partial\Phi}{\partial\xi}+\varepsilon\frac{g_{33}}{A_{55}}\left(\frac{\varphi_1+\varphi_2}{2h}\frac{\partial\Phi}{\partial\xi}+(\Phi+\theta(-\varphi_2))\frac{\partial\varphi_2}{h\partial\xi}\right)\\
&(x,y;\xi,\eta;13,23;5,4)\\
&\sigma_{zz}^{(1)}=\varepsilon\left(\frac{\varphi_1-\varphi_2}{2h}-\zeta\right)\left(\frac{\alpha_{13}}{A_{55}}\frac{\partial\Phi}{\partial\xi}+\frac{\alpha_{23}}{A_{44}}\frac{\partial\Phi}{\partial\eta}\right)+\varepsilon\frac{\varphi_1+\varphi_2}{2h}\left(\alpha_{13}+\frac{A_{13}}{A_{33}}\frac{\partial\Phi}{\partial\xi}+\alpha_{23}\frac{A_{23}}{A_{33}}\frac{\partial\Phi}{\partial\eta}\right)+\\
&+\varepsilon(\Phi+\theta(-\varphi_2))\left(\alpha_{13}\frac{A_{13}}{A_{33}}\frac{\partial\varphi_2}{h\partial\xi}+\alpha_{23}\frac{A_{23}}{A_{33}}\frac{\partial\varphi_2}{h\partial\eta}\right)\\
&\sigma_{xx}^{(1)}=A_{13}\sigma_{zz}^{(1)}+\varepsilon\left(\zeta+\frac{\varphi_2}{h}\right)\left(\alpha_{13}B_{11}\frac{\partial\Phi}{\partial\xi}+\alpha_{23}B_{12}\frac{\partial\Phi}{\partial\eta}\right)+\varepsilon(\Phi+\theta(-\varphi_2))\left(\alpha_{13}B_{11}\frac{\partial\varphi_2}{h\partial\xi}+\alpha_{23}B_{12}\frac{\partial\varphi_2}{h\partial\eta}\right)\\
&(x,y;A_{13},A_{23};B_{11},B_{12};B_{12},B_{22})\\
&\sigma_{xy}^{(1)}=\varepsilon B_{66}\left(\zeta+\frac{\varphi_2}{h}\right)\left(\alpha_{13}\frac{\partial\Phi}{\partial\eta}+\alpha_{23}\frac{\partial\Phi}{\partial\xi}\right)+\varepsilon(\Phi+\theta(-\varphi_2))\left(\alpha_{13}\frac{\partial\varphi_2}{h\partial\eta}+\alpha_{23}\frac{\partial\varphi_2}{h\partial\xi}\right)
\end{aligned}
\tag{2.6}
$$

Comparing (2.6) with (2.5), we can conclude that the order of the contribution of (2.6) to the general solution depends on the order of quantities of the type

$$
\frac{A_{13}}{A_{33}}g_{33}\frac{\partial\Phi}{\partial\xi},\quad \frac{A_{23}}{A_{33}}g_{33}\frac{\partial\Phi}{\partial\eta},\quad \frac{A_{13}}{A_{33}}\frac{\partial\varphi_2}{\partial\xi},\quad \frac{A_{23}}{A_{33}}\frac{\partial\varphi_2}{\partial\eta},\dots
$$

from which it follows that the combinations of ratios of the physical parameters and the variability of the characteristic function given in advance play an important role. If the terms in question are of order ε^{-1} as compared with the corresponding terms in the initial approximation, the above asymptotic form is unsuitable. In practical

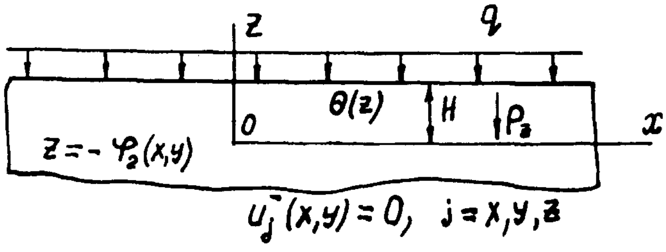

Fig. 3.

applications one can avoid such cases by an appropriate choice of the geometrical parameters and boundary functions.

When $\varphi_i(x, y) = \text{const}$ ($i = 1, 2$), i.e. for a layer of constant thickness, the solution (2.5) is exact. For arbitrary φ_i the iteration can be continued to obtain a solution with an accuracy specified in advance.

3. Consider a plate of variable thickness in a Cartesian system of coordinates x, y, z. Suppose that the upper face surface coincides with the plane $z = H$, which is acted upon by a uniformly distributed normal load with intensity q ($F_n = -q$), while the lower surface has an arbitrary form $z = -\varphi_1(x, y)$, with the components of the displacement vector specified on it. In particular, suppose that the surface is fixed to a rigid support ($u_x^- = u_y^- = u_z^- = 0$) (Fig. 3). The plate is subject to a temperature field $T - T_0 = \theta(z)$, and its own weight ($P_z = \rho g$) is taken into account. It is required to determine the stress–strain state. This is a typical problem when constructing foundations [15].

The solution of the problem with accuracy $O(\varepsilon)$ is given by

$$
\begin{aligned}
&\sigma_{xx} = -A_{13}(q+\rho gH) - K_1\rho gz - \lambda_1\theta \quad (xx, yy, xy; 1, 2, 6)\\
&\sigma_{zz} = -\rho g(H-z) - q, \quad \sigma_{xz} = \sigma_{yz} = 0\\
&u_x = u_x^- - (z+\varphi)A_{53}(q+\rho gH) - \lambda_5\bar{\theta} - \frac{1}{2}K_5\rho g(z^2 - \varphi^2)\\
&(x, y, z; 53, 43, 33; 5, 4, 3) \qquad\qquad (2.7)\\
&\bar{\theta} = \int_{-\varphi}^{z}\theta(z)dz, \quad K_i = a_{13}B_{1i} + a_{23}B_{2i} + a_{36}B_{6i}\\
&\lambda_i = \alpha_{11}B_{1i} + \alpha_{22}B_{2i} + \alpha_{12}B_{6i}, \quad i = 1, 2, 6\\
&K_j = a_{1j}K_1 + a_{2j}K_2 + a_{6j}K_6 - a_{3j}\\
&\lambda_j = a_{1j}\lambda_1 + a_{2j}\lambda_2 + a_{6j}\lambda_6 - \alpha_{(6-j)3}, \quad j = 3, 4, 5
\end{aligned}
$$

REFERENCES

1. GOL'DENVEIZER A. L., *The Theory of Thin Elastic Shells*. Nauka, Moscow, 1976.
2. GOL'DENVEIZER A. L., Construction of an approximate theory of plate bending by the method of asymptotic integration of the equations of the theory of elasticity. *Prikl. Mat. Mekh.* **26**, 4, 666–686, 1962.
3. FRIEDRICHS K. C. and DRESSLER R. F., A boundary-layer theory for elastic plates. *Comm. Pure Appl. Math.* **14**, 1, 1–33, 1961.
4. VOROVICH I. I., Some results and the problem of the asymptotic theory of plates and shells. In *Proceedings of the 1st All-Union School on the Theory and Numerical Methods of Computing Shells and Plates*. Izd. Tbil. Univ., Tbilisi, pp. 51–149, 1975.
5. AGALOVYAN L. A., On the theory of orthotropic plate bending. *Inzh. Zh. MTT* 6, 114–121, 1966.
6. AGALOVYAN L. A., On some relationships in the classical linear theory of anisotropic shells and the possibilities of refining them. *Izv. Akad. Nauk SSSR. MTT* 1, 109–120, 1972.
7. AGALOVYAN L. A., On the structure of the solution of a class of plane problems in the theory of elasticity for an anisotropic body. *Mekhanika, Izd. Yerevan. Univ., Yerevan* 2, 7–12, 1982.
8. AGALOVYAN L. A. and GEVORKYAN R. S., Non-classical boundary-value problems for plates with general anisotropy. In *Proceedings of the 4th All-Union Symposium on the Mechanics of Composite Material Structures*. Nauka, Novosibirsk, pp. 105–110, 1984.
9. AGALOVAN L. A. and GEVORKYAN R. S., On the asymptotic solution of non-classical boundary-value problems for two-layer anisotropic thermo-elastic shells. *Izv. Akad. Nauk Arm. SSR. Mekhanika* **42**, 3, 28–36, 1989.
10. AGALOVAN L. A., Determination of the stress–strain state of a two-layer strip and the validity of the Winkler hypothesis. In *Proceedings of the 13th All-Union Conference on the Theory of Plates and Shells*, Vol. 1. Izd. Politekhn. Inst., Tallinn, 13–18, 1983.
11. AGALOVAN L. A. and GEVORKYAN R. S., On the asymptotic solution of mixed three-dimensional problems for two-layer anisotropic plates. *Prikl. Mat. Mekh.* **50**, 271–278, 1986.

12. AGALOVAN L. A. and GEVORKYAN R. S., On non-classical boundary-value problems for three-layer thermo-elastic plates and some applications. In *Proceedings of the 14th All-Union Conference on the Theory of Plates and Shells*, Vol. 1. Izd. Tbil. Univ., Tbilisi, pp. 28–34, 1987.
13. AGALOVAN L. A., On the boundary layer of a plate. *Dokl. Akad. Nauk Arm. SSR* **55**, 3, 149–155, 1972.
14. GEVORKYAN R. S., The asymptotic behaviour of the boundary layer for a class of boundary-value problems for anisotropic plates. *Izv. Akad. Nauk Arm. SSR. Mekhanika* **37**, 6, 3–15, 1984.
15. VLASOV V. Z. and LEONT'YEV N. N., *Beams, Plates and Shells on an Elastic Foundation*. Fizmatgiz, Moscow, 1960.

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